A RESTRICTED SUM FORMULA FOR A q-ANALOGUE OF MULTIPLE ZETA VALUES

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Dedicated to Professor Michio Jimbo on his sixtieth birthday

ABSTRACT. We prove a new linear relation for a q-analogue of multiple zeta values. It is a q-extension of the restricted sum formula obtained by Eie, Liaw and Ong for multiple zeta values.

1. Introduction

Let $\alpha = (\alpha_1, \ldots, \alpha_r)$ be a multi-index of positive integers. We call the values r and $\sum_{i=1}^{r} \alpha_i$ *depth* and *weight* of α, respectively. If $\alpha_1 \geq 2$, we say that α is *admissible*. For an admissible index $(\alpha_1, \ldots, \alpha_r)$, *multiple zeta value* (MZV) is defined by

$$\zeta(\alpha_1, \ldots, \alpha_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{\alpha_1} \cdots m_r^{\alpha_r}}.$$

Let $I_0(r, n)$ be the set of admissible indices of depth r and weight n. In [3], Eie, Liaw and Ong proved the following relation called a restricted sum formula:

$$\sum_{\alpha \in I_0(b,n)} \zeta(\alpha_1, \ldots, \alpha_b, 1^a) = \sum_{\beta \in I_0(a+1, a+b+1)} \zeta(\beta_1 + n - b - 1, \beta_2, \ldots, \beta_{a+1}), \tag{1.1}$$

where $a \geq 0, b \geq 1, n \geq b+1$ and 1^a is an abbreviation of the subsequence $(1, \ldots, 1)$ of length a. It is a generalization of the sum formula proved in [4, 7], which is the equality (1.1) with $a = 0$.

In this paper we prove a q-analogue of the restricted sum formula. Let $0 < q < 1$. For an admissible index $\alpha = (\alpha_1, \ldots, \alpha_r)$, a *$q$-analogue of multiple zeta value* (qMZV) [1, 5, 8] is defined by

$$\zeta_q(\alpha_1, \ldots, \alpha_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{q^{(\alpha_1 - 1)m_1 + \cdots + (\alpha_r - 1)m_r}}{[m_1]^{\alpha_1} \cdots [m_r]^{\alpha_r}},$$

where $[n]$ is the q-integer $[n] := (1 - q^n)/(1 - q)$. In the limit $q \to 1$, qMZV converges to MZV. The main theorem of this article claims that qMZV's also satisfy the restricted sum formula:

Theorem 1.1. *For any integers $a \geq 0, b \geq 1$ and $n \geq b+1$, it holds that*

$$\sum_{\alpha \in I_0(b,n)} \zeta_q(\alpha_1, \ldots, \alpha_b, 1^a) = \sum_{\beta \in I_0(a+1, a+b+1)} \zeta_q(\beta_1 + n - b - 1, \beta_2, \ldots, \beta_{a+1}). \tag{1.2}$$

Setting $a=0$ we recover the sum formula for qMZV obtained by Bradley [2].

The strategy to prove Theorem 1.1 is similar to that of the proof for MZV's. However we should overcome some new difficulties. In the calculation of the q-analogue case, some additional terms are of the form $\sum_{n=1}^{\infty} q^{kn}/[n]^k$ $(k \in \mathbb{Z}_{\ge 1})$. In the limit of $q \to 1$, it becomes a harmonic sum $\sum 1/n^k$, but it is beyond a class of q-series described by qMZV's. To control such terms we make use of algebraic formulation of multiple harmonic series given in Section 2.2. We introduce a noncommutative polynomial algebra $\mathfrak{d}$ which is an extension of the algebra used in the proof of a q-analogue of Kawashima's relation for MZV [6]. Then the proof of Theorem 1.1 is reduced to an algebraic calculation in $\mathfrak{d}$ as will be seen in Section 2.3. We proceed the algebraic computation in Section 2.4 and finish the proof of Theorem 1.1.

Throughout this article we assume that $0<q<1$. We denote the set of multi-indices of positive integers, including non-admissible ones, of depth r and weight n by $I(r,n)$.

2. Proof

2.1. **Summation over indices.**

For $b \ge 1, n \ge 2$ and $M \in \mathbb{Z}_{\ge 1}$, define

$$K_{b,n}(M) := \sum_{\alpha \in I_0(b,n)} \sum_{m_1>m_2>\cdots>m_{b-1}>m_b=M} \prod_{j=1}^{b} \frac{q^{(\alpha_j-1)m_j}}{[m_j]^{\alpha_j}}.$$

Since $\alpha_1 \ge 2$, the infinite sum in the right hand side is convergent. Note that $K_{1,n}(M) = q^{(n-1)M}/[M]^n$.

For positive integers ℓ, β, M and N $(N>M)$, we set

$$f_\ell(N,M) := \sum_{N=k_1>k_2>\cdots>k_\ell>M} \frac{q^{k_1-M}}{[k_1-M]} \prod_{j=2}^{\ell} \frac{1}{[k_j-M]},$$

$$g_{\ell,\beta}(M) := \sum_{M=m_1 \ge m_2 \ge \cdots \ge m_\ell \ge 1} \frac{q^{(\beta-1)m_1}}{[m_1]^\beta} \prod_{j=2}^{\ell} \frac{q^{m_j}}{[m_j]}.$$

We set $f_\ell(N,M)=0$ unless $N>M$. Note that $g_{1,\beta}(M) = K_{1,\beta}(M)$.

Lemma 2.1. *For $M \ge 1$, $b \ge 1$ and $n \ge 2$, it holds that*

$$K_{b,n}(M) = g_{b,n-b+1}(M) - \sum_{s=1}^{b-1} \sum_{N=M+1}^{\infty} K_{b-s,n-s}(N) f_s(N,M).$$

Proof. For $k \geq 2$ and $m_1 > m_2$, it holds that

$$\begin{aligned}
&\sum_{\beta \in I(2,k)} \frac{q^{(\beta_1-1)m_1+(\beta_2-1)m_2}}{[m_1]^{\beta_1}[m_2]^{\beta_2}} \\
&= \frac{1}{[m_1][m_2]} \left(\left(\frac{q^{m_1}}{[m_1]} \right)^{k-1} - \left(\frac{q^{m_2}}{[m_2]} \right)^{k-1} \right) \Big/ \left(\frac{q^{m_1}}{[m_1]} - \frac{q^{m_2}}{[m_2]} \right) \\
&= \frac{q^{(k-2)m_2}}{[m_2]^{k-1}} \frac{1}{[m_1-m_2]} - \frac{q^{(k-2)m_1}}{[m_1]^{k-1}} \frac{q^{m_1-m_2}}{[m_1-m_2]}.
\end{aligned}$$

Using the above formula repeatedly we get

$$\begin{aligned}
K_{b,n}(M) &= \sum_{m_1 > \cdots > m_{b-1} > m_b = M} \frac{q^{m_1}}{[m_1]} \sum_{\beta \in I(b,n-1)} \prod_{j=1}^{b} \frac{q^{(\beta_j-1)m_j}}{[m_j]^{\beta_j}} \\
&= \sum_{m_1 > \cdots > m_{b-1} > m_b = M} \frac{q^{m_1}}{[m_1]} \left(\prod_{j=1}^{b-1} \frac{1}{[m_j - m_b]} \right) \frac{q^{(n-b-1)m_b}}{[m_b]^{n-b}} \\
&\quad - \sum_{s=1}^{b-1} \sum_{m_{b-s}=M+1}^{\infty} K_{b-s,n-s}(m_{b-s}) f_s(m_{b-s}, M).
\end{aligned}$$

The first term of the right hand side above is rewritten as follows. Setting $m_j = \ell_j + \cdots + \ell_{b-1} + M$ $(j = 1, \ldots, b-1)$, we have

$$\begin{aligned}
&\sum_{m_1 > \cdots > m_{b-1} > m_b = M} \frac{q^{m_1}}{[m_1]} \left(\prod_{j=1}^{b-1} \frac{1}{[m_j - m_b]} \right) \frac{q^{(n-b-1)m_b}}{[m_b]^{n-b}} \\
&= \frac{q^{(n-b-1)M}}{[M]^{n-b}} \sum_{\ell_1, \ldots, \ell_{b-1}=1}^{\infty} \frac{q^{\ell_1 + \cdots + \ell_{b-1} + M}}{[\ell_1 + \cdots + \ell_{b-1} + M]} \prod_{j=1}^{b-1} \frac{1}{[\ell_j + \cdots + \ell_{b-1}]}.
\end{aligned}$$

Now take the sum with respect to $\ell_1, \ell_2, \ldots, \ell_{b-1}$ successively using the equality

$$\sum_{\ell=1}^{\infty} \frac{q^{\ell+m}}{[\ell+m]} \frac{1}{[\ell+n]} = \sum_{\ell=1}^{\infty} \left(\frac{q^{\ell+n}}{[\ell+n]} - \frac{q^{\ell+m}}{[\ell+m]} \right) \frac{q^{m-n}}{[m-n]} = \frac{q^{m-n}}{[m-n]} \sum_{\ell=1}^{m-n} \frac{q^{\ell+n}}{[\ell+n]}$$

which holds for any $m > n$. Then we obtain $g_{b,n-b+1}(M)$. $\square$

Lemma 2.1 implies the following proposition, which can be proved by induction on b:

Proposition 2.2. *For positive integers r, ℓ and $N_1 > \cdots > N_r > M$, set*

$$h_{r,\ell}(N_1, \ldots, N_r, M) := \sum_{c \in I(r,\ell)} \left(\prod_{j=1}^{r-1} f_{c_j}(N_j, N_{j+1}) \right) f_{c_r}(N_r, M). \tag{2.1}$$

Then

$$K_{b,m}(M) = g_{b,m-b+1}(M) + \sum_{\ell=1}^{b-1}\sum_{r=1}^{\ell}(-1)^r \sum_{N_1>N_2>\cdots>N_r>M} g_{b-\ell,m-b+1}(N_1)\, h_{r,\ell}(N_1,\ldots,N_r,M) \tag{2.2}$$

for $b \geq 1, m \geq 2$ *and* $M \geq 1$.

Multiply $K_{b,m}(M)$ by the harmonic sum

$$\sum_{M>m_1>\cdots>m_a>0}\prod_{j=1}^{a}\frac{1}{[m_j]} \tag{2.3}$$

and take the sum over all $M \geq 1$. Then we get the left hand side of (1.2). In order to carry out the same calculation for the right hand side of (2.2), we prepare an algebraic formulation for multiple harmonic sums.

2.2. **Algebraic structure of multiple harmonic sums.** Denote by $\mathfrak{d}$ the non-commutative polynomial algebra over $\mathbb{Z}$ freely generated by the set of alphabets $S = \{z_k\}_{k=1}^{\infty} \cup \{\xi_k\}_{k=1}^{\infty}$. For a positive integer m, set

$$J_{z_k}(m) := \frac{q^{(k-1)m}}{[m]^k}, \qquad J_{\xi_k}(m) := \frac{q^{km}}{[m]^k}.$$

For a word $w = u_1 \cdots u_r \in \mathfrak{d}\, (r \geq 1, u_i \in S)$ and $M \in \mathbb{Z}_{\geq 1}$, set

$$A_w(M) := \sum_{M>m_1>\cdots>m_r>0} J_{u_1}(m_1)\cdots J_{u_r}(m_r),$$
$$A^{\star}_w(M) := \sum_{M>m_1\geq\cdots\geq m_r\geq 1} J_{u_1}(m_1)\cdots J_{u_r}(m_r).$$

We extend the maps $w \mapsto A_w(M)$ and $w \mapsto A^{\star}_w(M)$ to the $\mathbb{Z}$-module homomorphisms $A(M), A^{\star}(M) : \mathfrak{d} \to \mathbb{R}$ by $A_1(M) = 1, A^{\star}_1(M) = 1$ and $\mathbb{Z}$-linearity. Note that $A_{z_1^a}(M)$ is equal to the harmonic sum (2.3). If w is contained in the $\mathbb{Z}$-linear span of monomials $z_{i_1}\cdots z_{i_r}$ with $i_1 \geq 2$, $A_w(M)$ becomes a linear combination of qMZV's in the limit $M \to \infty$.

Denote by $\mathfrak{d}_{\xi}$ the $\mathbb{Z}$-subalgebra of $\mathfrak{d}$ generated by $\{\xi_k\}_{k=1}^{\infty}$. Define a $\mathbb{Z}$-bilinear map $\rho : \mathfrak{d}_{\xi} \times \mathfrak{d} \to \mathfrak{d}$ inductively by $\rho(1, w) = w\, (w \in \mathfrak{d})$, $\rho(v, 1) = v\, (v \in \mathfrak{d}_{\xi})$ and

$$\rho(\xi_k v, z_\ell w) = \xi_k\, \rho(v, z_\ell w) + z_\ell\, \rho(\xi_k v, w) + z_{k+\ell}\, \rho(v, w),$$
$$\rho(\xi_k v, \xi_\ell w) = \xi_k\, \rho(v, z_\ell w) + \xi_\ell\, \rho(\xi_k v, w) + \xi_{k+\ell}\, \rho(v, w)$$

for $v \in \mathfrak{d}_{\xi}$ and $w \in \mathfrak{d}$.

Proposition 2.3. *For* $v \in \mathfrak{d}_{\xi}, w \in \mathfrak{d}$ *and* $M \geq 1$, *we have* $A_v(M)A_w(M) = A_{\rho(v,w)}(M)$.

Proof. It is enough to consider the case where v and w are words. If $v = 1$ or $w = 1$, it is trivial. From the definition of $A(M)$, it holds that

$$A_{\xi_k w}(M) = \sum_{M>m>0} \frac{q^{km}}{[m]^k} A_w(m), \quad A_{z_\ell w}(M) = \sum_{M>n>0} \frac{q^{(\ell-1)n}}{[n]^\ell} A_w(n). \tag{2.4}$$

Hence we find

$$\begin{aligned} A_{\xi_k v}(M) A_{z_\ell w}(M) &= \left(\sum_{M>m>n>0} + \sum_{M>n>m>0} + \sum_{M>m=n>0} \right) \frac{q^{km}}{[m]^k} \frac{q^{(\ell-1)n}}{[n]^\ell} A_v(m) A_w(n) \\ &= \sum_{M>m>0} \frac{q^{km}}{[m]^k} A_v(m) A_{z_\ell w}(m) + \sum_{M>n>0} \frac{q^{(\ell-1)n}}{[n]^\ell} A_{\xi_k v}(n) A_w(n) \\ &+ \sum_{M>m>0} \frac{q^{(k+\ell-1)m}}{[m]^{k+\ell}} A_v(m) A_w(m) \end{aligned}$$

and a similar formula for $A_{\xi_k v}(M) A_{\xi_\ell w}(M)$. Now the proposition follows from the induction on the sum of length of v and w. □

For $k \geq 1$, we define a $\mathbb{Z}$-linear map $\xi_k \circ \cdot : \mathfrak{d}_\xi \to \mathfrak{d}_\xi$ inductively by $\xi_k \circ 1 = 0$ and $\xi_k \circ (\xi_\ell v) = \xi_{k+\ell} v$ for $v \in \mathfrak{d}_\xi$. Now consider the $\mathbb{Z}$-linear map $d : \mathfrak{d}_\xi \to \mathfrak{d}_\xi$ defined by $d(1) = 1$ and $d(\xi_k v) = \xi_k d(v) + \xi_k \circ d(v)$ $(v \in \mathfrak{d}_\xi)$.

Proposition 2.4. *For any $v \in \mathfrak{d}_\xi$ and $M \geq 1$, it holds that $A_v^\star(M) = A_{d(v)}(M)$.*

Proof. From the definition of $A(M)$ and $A^\star(M)$ we have

$$A^\star_{\xi_k v}(M) = \sum_{M>m>0} \frac{q^{km}}{[m]^k} A^\star_v(m+1), \qquad \sum_{M>m>0} \frac{q^{km}}{[m]^k} A_v(m+1) = A_{\xi_k v + \xi_k \circ v}(M).$$

To show the second formula, divide the sum $A_v(m+1)$ into the two parts with $m_1 = m$ and $m_1 < m$. Combining the two formulas above, we obtain the proposition by induction on length of v. □

2.3. **Algebraic formulation of the main theorem.** To calculate the right hand side of (2.2) multiplied by the harmonic sum (2.3), we need the following formula:

Lemma 2.5. *For $n_1 > \cdots > n_s > n_{s+1} > 0$, set*

$$p(n_1, \ldots, n_s; n_{s+1}) := \frac{q^{n_1 - n_{s+1}}}{[n_1 - n_{s+1}]} \prod_{j=2}^{s} \frac{1}{[n_j - n_{s+1}]}. \tag{2.5}$$

Let $s \geq 1$, $v = z_1$ or ξ_1, and N and M be positive integers such that $N > M$. Then it holds that

$$\begin{aligned}
&\sum_{N>n_1>\cdots>n_{s+1}>M} p(n_1, \ldots, n_s; n_{s+1}) J_v(n_{s+1}) \\
&= \sum_{N>k_1>\cdots>k_{s+1}>M} J_v(k_1) p(k_2, \ldots, k_s, k_{s+1}; M) \\
&+ \sum_{i=1}^{s} \sum_{N>k_1>\cdots>k_{s+1}>M} \frac{q^{k_1}}{[k_1]} \left(\prod_{j=2}^{i+1} \frac{1}{[k_j]} \right) p(k_{i+2}, \ldots, k_{s+1}; M),
\end{aligned}$$

where $p(\emptyset; M) = 1$.

Proof. Here we prove the lemma in the case of $v = z_1$. The proof for $v = \xi_1$ is similar. Using

$$\begin{aligned}
\frac{1}{[n_1 - n_{s+1}][n_{s+1}]} &= \frac{1}{[n_1 - n_{s+1}]} + \frac{q^{n_{s+1}}}{[n_{s+1}]}, \\
\frac{1}{[n_j - n_{s+1}][n_{s+1}]} &= \frac{q^{n_j - n_{s+1}}}{[n_j - n_{s+1}]} + \frac{1}{[n_{s+1}]} \quad (j = 2, \ldots, s),
\end{aligned}$$

we find that

$$\begin{aligned}
&p(n_1, \ldots, n_s; n_{s+1}) J_v(n_{s+1}) \\
&= \sum_{i=0}^{s} \frac{q^{(1-\delta_{i,0})n_1}}{[n_1]} \left(\prod_{j=2}^{i+1} \frac{1}{[n_j]} \right) p(n_{i+1}, \ldots, n_s; n_{s+1}).
\end{aligned}$$

Now take the sum of the both hand sides over $N > n_1 > \cdots > n_{s+1} > M$. In the right hand side, change the variables $n_1, \ldots, n_{s+1}$ to $k_1, \ldots, k_{s+1}$ by setting $n_t = k_t \, (1 \leq t \leq i+1)$, $n_t = k_{i+1} - k_{i+2} + k_{t+1} \, (i+2 \leq t \leq s)$ and $n_{s+1} = k_{i+1} - k_{i+2} + M$. Then we get the desired formula. $\square$

Let $\mathfrak{d}_1$ be the $\mathbb{Z}$-subalgebra of $\mathfrak{d}$ generated by z_1 and ξ_1. Motivated by Lemma 2.5 we introduce the $\mathbb{Z}$-module homomorphism $\varphi_s : \mathfrak{d}_1 \to \mathfrak{d}_1 \, (s \in \mathbb{Z}_{\geq 0})$ defined in the following way. Determine $\varphi_s(w)$ for a word $w \in \mathfrak{d}_1$ inductively on s and length of w by $\varphi_0 = \mathrm{id}$, $\varphi_s(1) = \xi_1 z_1^{s-1} \, (s \geq 1)$ and

$$\varphi_s(z_1 w) = z_1 \varphi_s(w) + \xi_1 \sum_{i=1}^{s} z_1^i \varphi_{s-i}(w), \qquad \varphi_s(\xi_1 w) = \xi_1 \sum_{i=0}^{s} z_1^i \varphi_{s-i}(w),$$

and extend it by $\mathbb{Z}$-linearity.

Proposition 2.6. *For $w \in \mathfrak{d}_1$ and any positive integers s, s', ℓ, β and N, we have*

$$\sum_{N>M_1>M_2>0} f_{s'}(N, M_1) f_s(M_1, M_2) A_w(M_2) = \sum_{N>M>0} f_{s'}(N, M) A_{\varphi_s(w)}(M), \tag{2.6}$$
$$\sum_{M_1>M_2>0} g_{\ell,\beta}(M_1) f_s(M_1, M_2) A_w(M_2) = \sum_{M>0} g_{\ell,\beta}(M) A_{\varphi_s(w)}(M).$$

Proof. Here we prove the first formula (2.6). The proof for the second is similar. It suffices to consider the case where $w = u_1 \cdots u_r\,(r \geq 1, u_i \in S)$ is a word. The left hand side of (2.6) is equal to

$$\sum f_{s'}(N, M_1)\, p(M_1, k_1, \ldots, k_{s-1}; M_2) \prod_{i=1}^{r} J_{u_i}(m_i),$$

where p is defined by (2.5) and the sum is over $M_1, k_i\,(1 \leq i \leq s-1), M_2, m_i\,(1 \leq i \leq r)$ with the condition $N > M_1 > k_1 > \cdots > k_{s-1} > M_2 > m_1 > \cdots > m_r > 0$. Changing the variables $(k_1, \ldots, k_{s-1}, M_2)$ to $(n_1, \ldots, n_s)$ by $k_i = M_1 - n_1 + n_{i+1}\,(1 \leq i \leq s-1)$ and $M_2 = M_1 - n_1 + m_1$, we obtain

$$\sum f_{s'}(N, M_1) p(n_1, \ldots, n_s; m_1) \prod_{i=1}^{r} J_{u_i}(m_i),$$

where the sum is over $N > M_1 > n_1 > \cdots > n_s > m_1 > \cdots > m_r > 0$. From Lemma 2.5 and the definition of φ_s, we see by induction on r that it is equal to the right hand side of (2.6). $\square$

We define the $\mathbb{Z}$-linear maps $\Phi_\ell : \mathfrak{d}_1 \to \mathfrak{d}_1\,(\ell \geq 0)$ by $\Phi_0 := \mathrm{id}$ and

$$\Phi_\ell := \sum_{r=1}^{\ell} (-1)^r \sum_{c \in I(r,\ell)} \varphi_{c_1} \cdots \varphi_{c_r},$$

and $Z_s : \mathfrak{d}_1 \to \mathfrak{d}\,(s \geq 0)$ by

$$Z_s(w) := \sum_{\ell=0}^{s} \rho(d(\xi_1^{s-\ell}), \Phi_\ell(w)).$$

Proposition 2.7. *For any integers $a \geq 0, b \geq 1$ and $n \geq b+1$, we have*

$$\sum_{\alpha \in I_0(b,n)} \zeta_q(\alpha_1, \ldots, \alpha_b, 1^a) = \sum_{s=0}^{b-1} \sum_{M>0} \frac{q^{(n-s-1)M}}{[M]^{n-s}} A_{Z_s(z_1^a)}(M). \tag{2.7}$$

Proof. Using Proposition 2.6 repeatedly, we have

$$\sum_{N_1>N_2>\cdots>N_r>M>0} g_{b,m}(N_1)\, h_{r,\ell}(N_1, \ldots, N_r, M) A_{z_1^a}(M)$$
$$= \sum_{c \in I(r,\ell)} \sum_{M>0} g_{b,m}(M) A_{\varphi_{c_1} \cdots \varphi_{c_r}(z_1^a)}(M),$$

where $h_{r,\ell}$ is defined by (2.1). Hence Proposition 2.2 implies that

$$\sum_{\alpha\in I_0(b,m)} \zeta_q(\alpha_1,\ldots,\alpha_b,1^a) = \sum_{M>0} K_{b,m}(M) A_{z_1^a}(M)$$
$$= \sum_{\ell=0}^{b-1}\sum_{M>0} g_{b-\ell,m-b+1}(M) A_{\Phi_\ell(z_1^a)}(M).$$

Substituting

$$g_{j,n}(M) = \sum_{t=0}^{j-1} \frac{q^{(n+j-t-2)M}}{[M]^{n+j-t-1}} A^{\star}_{\xi_1^t}(M),$$

we get the desired formula from Proposition 2.3 and Proposition 2.4. □

As we will see in the next subsection, the elements $Z_s(z_1^a)\,(s,a\ge 0)$ belong to the subalgebra of $\mathfrak{d}$ generated only by $\{z_k\}_{k=1}^{\infty}$ (see Proposition 2.10 and Proposition 2.14 below). Thus the right hand side of (2.7) will turn out to be a linear combination of qMZV's.

2.4. **Proof of the main theorem.** First we give a proof of Theorem 1.1 with $a=0$, that is, the sum formula for qMZV's. To this aim we prepare a recurrence relation of $d(\xi_1^k)\,(k\ge 0)$.

Lemma 2.8. *Let $k\ge 1$. Then*

$$d(\xi_1^k) = \sum_{r=1}^{k}\sum_{c\in I(r,k)} \xi_{c_1}\cdots\xi_{c_r}.$$

Proof. We prove the lemma by induction on k. The case of $k=1$ is trivial. Let $k\ge 2$. From the definition of d and the hypothesis of induction we see that

$$d(\xi_1^k) = d(\xi_1\cdot\xi_1^{k-1}) = \xi_1 \sum_{r=1}^{k-1}\sum_{c\in I(r,k-1)} \xi_{c_1}\cdots\xi_{c_r} + \xi_1\circ\left(\sum_{r=1}^{k-1}\sum_{c\in I(r,k-1)} \xi_{c_1}\cdots\xi_{c_r}\right)$$
$$= \sum_{r=2}^{k}\sum_{\substack{c\in I(r,k)\\ c_1=1}} \xi_{c_1}\cdots\xi_{c_r} + \sum_{r=1}^{k-1}\sum_{\substack{c\in I(r,k)\\ c_1\ge 2}} \xi_{c_1}\cdots\xi_{c_r} = \sum_{r=1}^{k}\sum_{c\in I(r,k)} \xi_{c_1}\cdots\xi_{c_r}.$$

This completes the proof. □

Corollary 2.9. *For $k\ge 1$ it holds that*

$$d(\xi_1^k) = \sum_{a=1}^{k} \xi_a d(\xi_1^{k-a}). \tag{2.8}$$

The sum formula for qMZV's follows from the following proposition.

Proposition 2.10. $Z_s(1)=\delta_{s,0}\,(s\ge 0)$.

Proof. Using $\Phi_\ell = -\sum_{a=1}^{\ell} \varphi_a \Phi_{\ell-a}\,(\ell \geq 1)$, we find that $\Phi_\ell(1) = (-\xi_1)^\ell\,(\ell \geq 0)$ by induction on ℓ. Thus the proposition is reduced to the proof of

$$\sum_{\ell=0}^{s} (-1)^\ell \rho(d(\xi_1^{s-\ell}), \xi_1^\ell) = \delta_{s,0}.$$

Let us prove it by induction on s. Denote the left hand side above by T_s. It is trivial that $T_0 = 1$. Let $s \geq 1$. Divide T_s into the three parts

$$T_s = d(\xi_1^s) + \sum_{\ell=1}^{s-1} (-1)^\ell \rho(d(\xi_1^{s-\ell}), \xi_1^\ell) + (-1)^s \xi_1^s.$$

Rewrite the second part by using (2.8) and the definition of ρ and d. Then we get

(2.9)
$$\sum_{a=1}^{s-1} \xi_a \sum_{\ell=1}^{s-a} (-1)^\ell \rho(d(\xi_1^{s-a-\ell}), \xi_1^\ell) - \sum_{\ell=0}^{s-2} (-1)^\ell \xi_1 \rho(d(\xi_1^{s-1-\ell}), \xi_1^\ell) - \sum_{a=1}^{s-1} \xi_{a+1} I_{s-a-1}.$$

From $(-1)^s \xi_1^s = -(-1)^{s-1} \xi_1 \rho(d(\xi_1^0), \xi_1^{s-1})$, which is the summand of the second term of (2.9) with $\ell = s-1$, and

$$d(\xi_1^s) = \sum_{a=1}^{s-1} \xi_a \rho(d(\xi_1^{s-a}), \xi_1^0) + \xi_s,$$

we obtain

$$T_s = \sum_{a=1}^{s-1} \xi_a T_{s-a} + \xi_s - \xi_1 T_{s-1} - \sum_{a=1}^{s-1} \xi_{a+1} T_{s-a-1}.$$

Therefore the induction hypothesis $T_a = \delta_{a,0}\,(a < s)$ implies that $T_s = 0$. $\square$

From Proposition 2.7 with $a = 0$ and Proposition 2.10, we see that

$$\sum_{\alpha \in I_0(b,n)} \zeta_q(\alpha_1, \ldots, \alpha_b) = \sum_{M>0} \frac{q^{(n-1)M}}{[M]^n} A_1(M) = \zeta_q(n).$$

Thus we get Theorem 1.1 in the case of $a = 0$. To complete the proof of Theorem 1.1, we should calculate $Z_s(z_1^a)$ for $a \geq 1$. For that purpose we prepare several lemmas.

Lemma 2.11. *For $\ell \geq 0$ and $w \in \mathfrak{d}_1$, it holds that*

$$\Phi_\ell(z_1 w) = \sum_{j=0}^{\ell} (-\xi_1)^{\ell-j} z_1 \Phi_j(w). \tag{2.10}$$

Proof. For non-negative integers a and n, set $\eta_{0,n} = \delta_{n,0}$ and

$$\eta_{a,n} := \sum_{c \in I(a,n)} \xi_1 z_1^{c_1} \cdots \xi_1 z_1^{c_a} \quad (a \geq 1).$$

Then it holds that

$$\varphi_s(\xi_1^a z_1 w) = \sum_{t=0}^{s} (\eta_{a,s-t} + \eta_{a+1,s-t-1}) z_1 \varphi_t(w),$$

where $\eta_{a+1,-1} := 0$, for $a \ge 0, s \ge 0$ and $w \in \mathfrak{d}_1$. Using this formula we prove (2.10) by induction on ℓ. The case of $\ell = 0$ is trivial. Let $\ell \ge 1$. The induction hypothesis and the relation $\Phi_\ell = -\sum_{a=1}^{\ell} \varphi_a \Phi_{\ell-a}$ imply that

$$\Phi_\ell(z_1 w) = \sum_{j=0}^{\ell-1} \sum_{a=1}^{\ell-j} \sum_{t=0}^{a} (\eta_{\ell-a-j,a-t} + \eta_{\ell-a-j+1,a-t-1}) z_1 \varphi_t(\Phi_j(w)).$$

Divide the sum into the two parts with $t = 0$ and $t \ge 1$, and take the sum with respect to a. Then we obtain

$$\sum_{j=0}^{\ell-1} \left\{ (-\delta_{\ell-j,0} + (-1)^{\ell-j} \eta_{\ell-j,0}) z_1 \varphi_0(\Phi_j(w)) - \sum_{t=1}^{\ell-j} \delta_{\ell-j-t,0} z_1 \varphi_t(\Phi_j(w)) \right\}.$$

Since $\eta_{\ell-j,0} = \xi_1^{\ell-j}$, $\varphi_0 = \mathrm{id}$ and $-\sum_{j=0}^{\ell-1} \varphi_{\ell-j} \Phi_j = \varphi_\ell$, it is equal to the right hand side of (2.10). ∎

Lemma 2.12. *For $k \ge 0$ and $w \in \mathfrak{d}_1$, it holds that*

$$\sum_{\ell=0}^{k} \rho(d(\xi_1^{k-\ell}), \xi_1^{\ell} z_1 w) = \sum_{\ell=0}^{k} z_{\ell+1}\, \rho(d(\xi_1^{k-\ell}), w). \tag{2.11}$$

Proof. Denote the left hand side and the right hand side of (2.11) by L_k and R_k, respectively. The equality (2.11) holds when $k = 0$ because $L_0 = \rho(1, z_1 w) = z_1 w = z_1 \rho(1, w) = R_0$. Hereafter we assume that $k \ge 1$.

Divide L_k into the three parts

$$L_k = \rho(d(\xi_1^k), z_1 w) + \sum_{\ell=1}^{k-1} (-1)^\ell \rho(d(\xi_1^{k-\ell}), \xi_1^\ell z_1 w) + (-1)^k \xi_1^k z_1 w. \tag{2.12}$$

Let us rewrite the first part. Substitute (2.8) into $d(\xi_1^k)$. From the definition of ρ we see that the first part is equal to

$$\sum_{a=1}^{k} \left(\xi_a \rho(d(\xi_1^{k-a}), z_1 w) + z_1 \rho(\xi_a d(\xi_1^{k-a}), w) + z_{a+1} \rho(d(\xi_1^{k-a}), w) \right).$$

Note that the first term of the summand with $a = k$ is equal to $\xi_k z_1 w = \xi_k L_0$. Apply (2.8) again to the second term, and we see that the first part of the right hand side of (2.12) is equal to

$$\xi_k L_0 + \sum_{a=1}^{k-1} \xi_a \rho(d(\xi_1^{k-a}), z_1 w) + R_k. \tag{2.13}$$

We proceed the same calculation for the second part of (2.12). Here we decompose $\xi_1^\ell z_1 w = \xi_1 \cdot \xi_1^{\ell-1} z_1 w$ and use (2.8). As a result we get

$$\sum_{a=1}^{k-1}\sum_{\ell=1}^{k-a}(-1)^\ell \xi_a \rho(d(\xi_1^{k-\ell-a}), \xi_1^\ell z_1 w) - \sum_{a=1}^{k-1}\xi_{a+1} I_{k-a-1} \\ - \sum_{\ell=0}^{k-2}(-1)^l \xi_1 \rho(d(\xi_1^{k-1-\ell}), \xi_1^\ell z_1 w). \tag{2.14}$$

Note that the third part of (2.12) is equal to

$$-(-1)^{k-1}\xi_1\rho(d(\xi_1^0), \xi_1^{k-1} z_1 w), \tag{2.15}$$

which is the summand of the third term of (2.14) with $\ell = k-1$. Hence the three parts (2.13), (2.14) and (2.15) add up to

$$\xi_k L_0 + \sum_{a=1}^{k-1}\xi_a L_{k-a} + R_k - \sum_{a=1}^{k-1}\xi_{a+1} L_{k-a-1} - \xi_1 L_{k-1} = R_k.$$

This completes the proof. □

Now we can prove the key formula to calculate $Z_s(z_1^a)$ for $a \geq 1$:

Proposition 2.13. *Let $w \in \mathfrak{d}_1$ and $s \geq 0$. Then $Z_s(z_1 w) = \sum_{\ell=0}^{s} z_{\ell+1} Z_{s-\ell}(w)$.*

Proof. Using (2.10) we have

$$Z_s(z_1 w) = \sum_{\ell=0}^{s}\sum_{j=0}^{l}(-1)^{\ell-j}\rho(d(\xi_1^{s-\ell}), \xi_1^{\ell-j} z_1 \Phi_j(w)).$$

Because of Lemma 2.12 it is equal to

$$\sum_{j=0}^{s}\sum_{\ell=0}^{s-j} z_{\ell+1}\rho(d(\xi_1^{s-j-\ell}), \Phi_j(w)) = \sum_{\ell=0}^{s} z_{\ell+1} Z_{s-l}(w).$$

This completes the proof. □

Combining Proposition 2.10 and Proposition 2.13, we obtain the following formula:

Proposition 2.14. *For $s \geq 0$ and $a \geq 1$, it holds that*

$$Z_s(z_1^a) = \sum_{\gamma \in I(a, s+a)} z_{\gamma_1} \cdots z_{\gamma_a}.$$

At last let us prove Theorem 1.1 in the case of $a \geq 1$. From Proposition 2.7 and Proposition 2.14, it holds that

$$\sum_{\alpha \in I_0(b,n)} \zeta_q(\alpha_1, \ldots, \alpha_b, 1^a) = \sum_{s=0}^{b-1}\sum_{\gamma \in I(a, s+a)} \zeta_q(n-s-1, \gamma_1, \ldots, \gamma_a).$$

Set $\beta_1 = b+1-s$. The right hand side becomes

$$\sum_{\beta_1=2}^{b+1} \sum_{\gamma \in I(a,a+b+1-\beta_1)} \zeta_q(\beta_1+n-b-1, \gamma_1, \ldots, \gamma_a)$$
$$= \sum_{\beta \in I_0(a+1,a+b+1)} \zeta_q(\beta_1+n-b-1, \beta_2, \ldots, \beta_{a+1}).$$

This completes the proof of Theorem 1.1.

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